INTERACTION OF BRIGHT AND DARK SOLITARY WAVES IN A MULTICOMPONENT PLASMA WITH SUPERThERMAL ELECTRONS FOR DIFFERENT COORDINATES SYSTEMS

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ABSTRACT
The interaction behavior of planar/nonplanar of bright and dark ion acoustic solitary waves (IASWs) in a multicomponent plasma composed of positive and negative ion fluids and kappa distributed superthermal electrons is investigated. For the generic case, the extended Poincaré-Lighthill-Kuo (PLK) method is employed to derive planar and nonplanar Korteweg-de Vries (KdV) equations and phase shift equations. The nonlinear propagation and the collision of planar/nonplanar bright and dark IASWs are studied. On the other hand, at a critical value of ion concentration, the extended PLK method is applied to obtain the modified planar/nonplanar KdV equations and their corresponding phase shifts. It is found that planar/nonplanar bright – planar/nonplanar dark IASWs can propagate and collide with each other. In all cases, the effects of several parameters such as: ion concentration, the position of IASWs, negative ion-to-positive ion mass ratio, the superthermality of electrons on the trajectories of planar/nonplanar bright and dark IASWs after collision are investigated. Numerical calculations lead to some highlights on the properties of IASWs (e.g., in laboratory plasmas such as laser–matter/plasma interaction experiments).

1. INTRODUCTION
The nonlinear planar/nonplanar ion acoustic solitary waves (IASWs), as one important nonlinear phenomena in a plasma, have been studied by many authors in various models of plasma physics in recent years [1-4]. Observation of ion acoustic solitons experimentally for first time [5], and in the years that followed, several studies of ion acoustic solitary waves have been discussed [6-8]. Furthermore, IASWs in multicomponent plasma have been studied by a number of authors resulting in a considerable success in clarifying many aspects of the characteristics of solitary planar [9, 10] and nonplanar systems [11-14]. An unmagnetized plasma consisting of warm adiabatic ions, superthermal electrons, and thermal positrons were considered [15]. This study may introduce a focus on the properties of ion acoustic solitary
waves in space and laboratory plasmas. Several theoretical and experimental investigations report the importance of study of suprathermal in non-Maxwellian plasma particles which arise due to the influence of external fields acting on the space or laboratory plasmas (represented by a Kappa distribution) [16-20]. For example, the head-on collision of two concentric cylindrical ion acoustic solitary waves is discussed [21]. They showed new effects, geometric and dynamics which are absent in Cartesian solitons and could be tested in experimental nonlinear ion acoustic waves in plasma. Also, the head-on collision between two ion-acoustic solitary waves in an unmagnetized electron-positron-ion plasma has been investigated [22]. The effects of the ratio of electron temperature to positron temperature, and the ratio of the number density of positrons to that of electrons on the phase shift are studied. It is found that these parameters can significantly influence the phase shifts of the solitons and the compressive solitary wave can propagate in this system. Head-on collision of ion acoustic solitary waves in a three-component unmagnetized plasma with cold ions, Boltzmann distributed positrons, and superthermal electrons [23]. The effects of several plasma parameters such as: the ratio of electron temperature to positron temperature, the spectral index of the electron kappa distribution, and fractional concentration of positron component on the phase shift are studied. It is found that the superthermal electrons play a significant role on the collision of ion acoustic solitary waves. Continually, the nonlinear propagation of small but finite amplitude ion acoustic wave with superthermal electrons and positrons in a bi-ion collisionless plasma are discussed [24]. They have examined the effect of the electron superthermality on the ion acoustic soliton characteristics and insure supporting of their model to compressive as well as rarefactive solitary structures. An investigation to the head-on collision of cylindrical/spherical ion-acoustic solitary waves in an unmagnetized non-planar plasma consisting of warm adiabatic ions and nonthermally distributed electrons [25]. They found that the phase shifts induced by the collision of compressive and rarefactive solitary waves are very different. They also pointed out to the importance of these investigations about the observations of electrostatic solitary structures in astrophysical as well as in experimental plasmas with nonthermal energetic electrons.

The extended Poincaré - Lighthill - Kuo (PLK) method have been used to study the propagation and interaction between two solitary waves [26-32]. Great number of others studies the general case of solitary waves interaction by deriving the Korteweg-de Vries (KdV) equations and the corresponding phase shifts, and didn't mention the critical case of interaction. The one dimensional planar geometry and solitons interaction by deducing the modified KdV (mKdV) equations and their
corresponding phase shifts at the critical case [33–35]. The moment, there is no detailed study for the difference between planar and nonplanar IASWs interaction in general and critical case. Thus, the purpose of this article is to study and compare the effect of different plasma parameters; negative ion-to-positive ion mass ratio, superthermal electron parameter on the behavior of planar and nonplanar ion acoustic solitary waves (IASWs) interactions in plasma. We organized this paper as follows. Sec. II includes the basic equations for planar and nonplanar IASWs including positive and negative ions and superthermal electrons. For generic case, the PLK method is introduced and the planar and nonplanar Korteweg–de Vries (KdV) equations are derived in order to describe the propagation of ion acoustic solitons, and in Sec. III we continue their corresponding phase shift. Furthermore, a modified planar and nonplanar Korteweg–de Vries (mKdV) equations with their corresponding phase shifts during interaction have been discussed at a critical case, Sec. IV. Numerical results and discussion are given in Sec. V.

II. THE GOVERNING EQUATIONS

We consider two fluids composed of positive and negative ions distinguished by using the index “+” and “−”, and superthermal electrons. The planar and nonplanar dynamics of IASWs in multicomponent plasma are governed by [36]

\[
\frac{\partial n_\pm}{\partial t} + \frac{\nu}{r} (n_\pm u_\pm) + \frac{\partial}{\partial r} (n_\pm u_\pm) = 0, \tag{1}
\]

\[
\frac{\partial u_\pm}{\partial t} + u_\pm \frac{\partial u_\pm}{\partial r} = \beta_\pm \frac{\partial \phi}{\partial r}, \tag{2}
\]

\[
\frac{\nu \partial \phi}{r \partial r} + \frac{\partial^2 \phi}{\partial r^2} = \mu_1 n_e + \mu_2 n_+ - n_-, \tag{3}
\]

where \( \nu = 0 \) for planar geometry factor and \( \nu = 1 \) and 2 for cylindrical and spherical (i.e., nonplanar) geometries factors, respectively. The variable \( n_\pm, n_\pm \) is the density of positive (negative) ions, \( u_\pm, (u_\pm) \) is the velocity of positive (negative) ions, \( \phi, r, \) and \( t \) is the electrostatic potential, the space coordinate and time variables, respectively. In Eqs. 1-3, the ion density \( n_\pm, n_\pm \) is normalized by the unperturbed ion density \( n_\pm^0, (n_\pm^0) \), while \( u_\pm, (u_\pm) \) is normalized by the ion sound speed \( c_s = \sqrt{(K_B T_e)/m_+} \). The variables \( n_e, \phi \) are respectively normalized by the unperturbed electron density \( n_e^0 \) and the thermal potential \( (K_B T_e/e Z_+) \). The space coordinate is normalized by the Debye radius \( \lambda_D = (K_B T_e / 4\pi n_e e^2 Z_+ + n_o)^{1/2} \) and the time is scaled by the ion plasma period \( \omega_{pi}^{-1} = (4\pi n_e e^2 Z_+ n_o / m_+)^{-1/2} \), where \( K_B \) is the Boltzmann constant and \( T_e \) is the
electron temperature. Here, \(m_+\) (\(m_-\)) is the positive (negative) ion mass and the positive ions charge \(Z_+=1\). The dispersion relation implies \(\mu_1+\mu_2=1\), where \(\mu_1(=n_{e0}/n_{+0})\) and \(\mu_2(=n_{-0}/n_{+0})\).

The electrons number density \(n_e\) are assumed to possess superthermal distribution \([36,37]\)

\[
n_e = \left(1 - \frac{\varphi}{\chi} \right)^{-\kappa + \frac{1}{2}},
\]

which only valid for \(\kappa > 3/2\), where \(\kappa\) is the superthermal parameter and \(\chi = (\kappa - \frac{3}{2})\). The fluid equations are coupled through the Poisson’s equation. We apply the extended PLK method to study planar and nonplanar IASWs interaction, so that, the following stretching is introduced \([19,29]\)

\[
\xi = \varepsilon(r - \lambda t - r_1) + \varepsilon^2 P_o(\xi, \eta, R) + \ldots, \\
\eta = \varepsilon(r + \lambda t - r_2) + \varepsilon^2 Q_o(\xi, \eta, R) + \ldots, \\
R = \varepsilon^3 r.
\]

The quantities \(n_\pm, u_\pm\) and \(\varphi\) can be expanded about their equilibrium values in power series of \(\varepsilon\) as follows \([21,38]\)

\[
n_\pm = 1 + \varepsilon n_\pm^{(1)} + \varepsilon^2 n_\pm^{(2)} + \varepsilon^3 n_\pm^{(3)} + \ldots,
\]

\[
u_\pm = \varepsilon u_\pm^{(1)} + \varepsilon^2 u_\pm^{(2)} + \varepsilon^3 u_\pm^{(3)} + \ldots, \\
\varphi = \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3 + \ldots.
\]

We introduce the following operators:

\[
\mathbf{\hat{R}} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right), \\
\mathbf{\hat{\hat{R}}} = \left\{\frac{\partial}{\partial \mathbf{\hat{R}}} + \left(P_o + P_\eta \right) \frac{\partial}{\partial \xi} + \left(Q_o + Q_\eta \right) \frac{\partial}{\partial \eta}\right\},
\]

\[
\mathbf{\hat{T}} = \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right), \\
\mathbf{\hat{\hat{T}}} = \lambda \left\{\left(P_\eta - P_o \right) \frac{\partial}{\partial \xi} + \left(Q_\eta - Q_o \right) \frac{\partial}{\partial \eta}\right\}.
\]

One can write these operators in a compact way, viz.

\[
\frac{\partial}{\partial r} = \varepsilon \mathbf{\hat{R}} + \varepsilon^3 \mathbf{\hat{\hat{R}}} + \ldots, \\
\frac{\partial}{\partial t} = \varepsilon \lambda \mathbf{\hat{T}} + \varepsilon^3 \mathbf{\hat{\hat{T}}} + \ldots,
\]

To lowest nonzero order Eqs. 1–5 give

\[
\lambda T n_+^{(1)} + R u_+^{(1)} = 0, \\
\lambda T n_-^{(1)} + R u_-^{(1)} = 0, \\
\lambda T u_+ + R \varphi_1 = 0,
\]

\[
\lambda T u_-^{(1)} - \beta R \varphi_1 = 0, \\
\mu_1 \frac{\kappa - \frac{3}{2}}{\chi} \varphi_1 + \mu_2 n_+^{(1)} =
\]
The solutions of Eqs. 7 can be given as

\[
\begin{align*}
\varphi_1 &= (\varphi_{1\xi} + \varphi_{1\eta}), &(8a) & u_+^{(1)} &= \frac{i}{\lambda}(\varphi_{1\xi} - \varphi_{1\eta}),
\end{align*}
\]

\[
\begin{align*}
n_+^{(1)} &= \frac{i}{\lambda^2}(\varphi_{1\xi} + \varphi_{1\eta}), &(8b) & u_-^{(1)} &= \frac{-\beta}{\lambda}(\varphi_{1\xi} - \varphi_{1\eta}),
\end{align*}
\]

\[
\begin{align*}
n^{(3)} &= -\frac{\beta}{\lambda^2}(\varphi_{1\xi} + \varphi_{1\eta}), & (8c) & \lambda &= \sqrt{\chi(\mu_2\beta + 1)/\mu_1(\kappa - 1/2)}.
\end{align*}
\]

After some algebraic manipulation we the next order in \( \varepsilon \) as

\[
\begin{align*}
\lambda \bar{T}_n^{(2)} + \bar{R}u_+^{(2)} + \bar{R}n_+^{(1)}u_+^{(1)} &= 0, & (9a) & \lambda \bar{T}u_-^{(2)} - \beta \bar{R}\varphi_2 + \bar{R}n_-^{(1)}u_-^{(1)} &= 0, & (9d)
\end{align*}
\]

\[
\begin{align*}
\lambda \bar{T}_n^{(2)} + \bar{R}u_-^{(2)} + \bar{R}n_-^{(1)}u_-^{(1)} &= 0, & (9b) & \mu_1 \frac{\kappa - 1}{\chi} \varphi_2 + \mu_2 n_-^{(2)} + \\
\lambda \bar{T}u_+^{(2)} + \bar{R}\varphi_2 + \bar{R}u_+^{(1)} &= 0, & (9c) & \mu_1 \frac{(\kappa - 1)(\kappa - \frac{3}{2})}{2\chi^2} \varphi_1^2 - n_+^{(2)} &= 0.
\end{align*}
\]

By solving this system of Eqs we have

\[
\begin{align*}
\varphi_2 &= (\varphi_{2\xi} + \varphi_{2\eta} + \bar{\varphi}_2), & (10a) & u_+^{(2)} &= \frac{i}{\lambda}(\varphi_{2\xi} - \varphi_{2\eta}) + \\
&+ \frac{3}{2\lambda^4}(\varphi_{1\xi}^2 + \varphi_{1\eta}^2) + \bar{n}_+, & (10b) & u_-^{(2)} &= \frac{-\beta}{\lambda}(\varphi_{2\xi} - \varphi_{2\eta}) + \\
&+ \frac{\beta^2}{2\chi^2}(\varphi_{1\xi}^2 - \varphi_{1\eta}^2) + \bar{n}_-.
\end{align*}
\]

\[
\begin{align*}
n_+^{(2)} &= \frac{1}{\lambda^2}(\varphi_{2\xi} + \varphi_{2\eta}) & & (10c) & 3\frac{\beta^2}{2\lambda^4}(\varphi_{1\xi}^2 + \varphi_{1\eta}^2) + \bar{n}_-.
\end{align*}
\]

Here \( \bar{\varphi}_2 = \varphi_2 - \varphi_{2\xi} - \varphi_{2\eta} \) has been defined as the part of \( \varphi_2 \) which depends on \( \xi, \eta \) and \( \tau \) in a way which cannot be separated with similar definitions for other variables and higher orders [34].

The expressions in Eqs. 10 are coupled through

\[
\delta_1 \varphi_{2\xi}^2 = 0, & & (11a) & \delta_1 \varphi_{2\eta}^2 = 0,
\]

and
where \( \delta_1 = \left( \frac{\mu_1(\kappa-\lambda)}{\chi^2} - \frac{3(1-\mu_2\beta^2)}{\lambda^4} \right) \), and \( \delta_2 = \frac{(1-\mu_2\beta^2)}{\lambda^4(1+\mu_2\beta)} \).

It is clear that from Eq. 11, there exist two possibilities; one is called generic case, at \( \phi_{1\xi} = \phi_{1\eta} = 0 \), the other one is called critical case, at \( \delta_1 = 0 \).

III. GENERIC CASE:

The generic case can be achieved at \( \phi_{1\xi} = \phi_{1\eta} = 0 \) and thus find that the series expansion go in even orders of \( \varepsilon \) much as is the case for the usual derivation of the KdV equation.

Finally, we arrive at the order to a set of equations where interesting new contributions appear:

\[
\lambda \bar{T}_{n+}^{(4)} + \bar{T}_{n+}^{(2)} + \frac{\nu}{R} \bar{u}_{n+}^{(2)} + \beta \bar{R} \bar{u}_{n+}^{(2)} = 0, \quad \beta \bar{R} \phi_2 = 0, \quad (12a)
\]

\[
\lambda \bar{T}_{u+}^{(4)} + \bar{T}_{u+}^{(2)} + \frac{\nu}{R} u_{u+}^{(2)} + \beta R u_{u+}^{(2)} = 0, \quad (12b)
\]

\[
\frac{\delta_1}{\delta_2} \phi_2 - \mu_1 \frac{\kappa-\lambda}{\chi} \phi_4 - \mu_2 n_{-}^{(4)} + n_{+}^{(4)} = 0, \quad (12c)
\]

\[
+ \bar{R} \phi_4 + \bar{R} \phi_2 = 0, \quad (12c)
\]

Now, combining the parts of Eq. 12 by using Eq. 10 which contain terms that only depend on \( \xi \) or on \( \eta \) (besides \( \tau \)), yields the typical cylindrical Korteweg-de Vries (cKdV) equations

\[
\frac{\partial \phi_{2\xi}}{\partial R} + A \phi_{2\xi} \frac{\partial \phi_{2\xi}}{\partial \xi} + B \frac{\partial^3 \phi_{2\xi}}{\partial \xi^3} + \frac{\nu}{2R} \phi_{2\xi} = 0, \quad (13)
\]

\[
\frac{\partial \phi_{2\eta}}{\partial R} + A \phi_{2\eta} \frac{\partial \phi_{2\eta}}{\partial \eta} + B \frac{\partial^3 \phi_{2\eta}}{\partial \eta^3} + \frac{\nu}{2R} \phi_{2\eta} = 0, \quad (14)
\]

for the right- and left-going solitary waves, respectively, with the coefficient of the quadratic nonlinearity, \( A \), and the dispersion coefficient \( B \) are given by \( -B \delta_1 \), and \( \frac{\delta_1^2}{2(\mu_2\beta+1)} \), respectively.

Furthermore, there is more information still, in the terms which contain both \( \xi \) and \( \eta \) besides \( \tau \), giving
The second and third terms in Eq. 15 will generate secular contributions at the next higher order therefore we have:

\[
\frac{\partial P_0}{\partial \eta} = S \phi_{2\eta}, \quad (17a) \quad \frac{\partial Q_0}{\partial \xi} = S \phi_{2\xi}, \quad (17b)
\]

In addition, the other remaining term in 15 will give rise to a contribution \(\bar{\phi}_4 = S_1 \phi_2 \phi_{2\eta}\), to \(\phi_4, \phi_{2\xi}\) and \(\phi_{2\eta}\) parts, which will have to be determined from higher-order contributions.

For large \(R\), the asymptotic solitary wave solutions of Eqs. 12 and Eqs. 13 these are the well-known "sech squared" solitons of KdV theory, here [39-41]}

\[
\phi_{2\xi} = \phi_\xi \theta_1^{2\nu/3} \text{sech}^2 \left( \frac{A \phi_\xi}{12B} \theta_1^{1/3} \left( \xi - \frac{1}{3} A \phi_\xi \theta_1^{2\nu/3} R \right) \right), \quad (19)
\]

\[
\phi_{2\eta} = \phi_\eta \theta_2^{2\nu/3} \text{sech}^2 \left( \frac{A \phi_\eta}{12B} \theta_2^{1/3} \left( \eta - \frac{1}{3} A \phi_\eta \theta_2^{2\nu/3} R \right) \right), \quad (20)
\]

where \(\phi_\xi\) and \(\phi_\eta\) are the amplitudes of solitons \(S_1\) and \(S_2\) in their initial positions. Here, the value of \(\theta_1\) and \(\theta_2\) are given by \(R_1/R\), \(R_2/R\), respectively, where \(R_1=ε^3 r_1\) and \(R_2=ε^3 r_2\). Now, the phase shift functions \(P_0(\xi, \eta, R)\), and \(Q_0(\xi, \eta, R)\) are obtained by substituting for \(\phi_{2\xi}\) and \(\phi_{2\eta}\) into Eq. 17, we have

\[
P_0 = S \left( \frac{12B \phi_\eta}{A} \right)^{1/2} \theta_2^{\nu/3} \left\{ \tanh \left[ \frac{A \phi_\eta}{12B} \theta_2^{1/3} \left( \eta - \frac{1}{3} A \phi_\eta \theta_2^{2\nu/3} R \right) \right] + 1 \right\}, \quad (21)
\]
Q_0 = S \left( \frac{12B \phi \xi}{A} \right)^{1/2} \theta_1^{v/3} \left\{ \tanh \left( \frac{\xi}{12B} \right)^{1/2} \theta_1^{v/3} \left( \frac{1}{3} A \phi \theta_1^{2v/3} R \right) - 1 \right\}, \quad (22)

where \( \xi_{t=0} = \varepsilon(r_2 - r_1), \eta_{t=0} = \varepsilon(r_1 - r_2), \xi_t = \varepsilon[(r_2 - r_1) - 2\lambda t] \) and \( \eta_t = \varepsilon[(r_1 - r_2) + 2\lambda t] \). So that, the trajectories of the two IASWs are given by

\[
\xi = \varepsilon(r - \lambda t - r_1) - \varepsilon^2 S \left( \frac{12B \phi \eta}{A} \right)^{1/2} \theta_2^{v/3} \left\{ \tanh \left( \frac{\xi}{12B} \right)^{1/2} \theta_2^{v/3} \left( \eta - \frac{1}{3} A \phi \eta^{2v/3} R \right) + 1 \right\}.
\]

\[
\eta = \varepsilon(r + \lambda t - r_2) - \varepsilon^2 S \left( \frac{12B \phi \xi}{A} \right)^{1/2} \theta_1^{v/3} \left\{ \tanh \left( \frac{\eta}{12B} \right)^{1/2} \theta_1^{v/3} \left( \xi - \frac{1}{3} A \phi \xi^{2v/3} R \right) - 1 \right\}.
\]

Using Eqs. 23 and Eqs. 24, the corresponding phase shifts are given by

\[\Delta P = 2\varepsilon^2 S \left( \frac{12B \phi \eta}{A} \right)^{1/2} \theta_2^{v/3},\]

\[\Delta Q = -2\varepsilon^2 S \left( \frac{12B \phi \xi}{A} \right)^{1/2} \theta_1^{v/3},\]

Fig. 1. The variation of nonlinear coefficient A, against 1/\( \mu_2 \) for \( \kappa = 1.8 \), for planar and nonplanar geometry at \( \phi_{2\xi} = 0.9, \phi_{2\eta} = 0.3, \varepsilon = 0.1, \phi_n = \)
In this part of the present manuscript, we employed the extended PLK method to drive the planar and nonplanar Korteweg-de Vries (KdV) equations and phase shift equations are derived to investigate the future of IASWs after collision. Actually, it is valuable here mentioning that there exist two interacting solitons must have the same polarity given by the sign of AB. If AB > 0, both solitons must have positive polarity, but when AB < 0, both solitons must have negative polarity. Figure 1 represents the relation between the amplitude of IASWs and the negative ion-to-positive ion density ratio $\mu_2$. The ranges for the soliton polarities shown in Fig.1 agree with those found in recent head-on collisions of electrostatic solitons [33, 34]. Moreover, we consider two soliton $S_1$ is traveling to the right and soliton $S_2$ is traveling to the left, we see from Eqs. 25 and Eqs. 26 that due to collision, each soliton has a positive phase shift in its traveling direction. Physically, the positive phase shift means that IASWs increase their velocities during the collision stage [42].

![Figure 2](image.png)

**FIG. 2.** The variation of the phase shift $\Delta P$ with $\mu_1$ for planar geometry ($\nu = 0$), cylindrical geometry ($\nu = 1$) and for spherical geometry ($\nu = 2$) at $\phi_2 = 0.9$, $\phi_1 = 0.3$, $\nu = 0.1$, $\phi_2 = 0.1$, $\phi_1 = 0.4$.

Figure 2 explores the differences in the phase shifts $\Delta P$ for planar and nonplanar (cylindrical/spherical) geometries. It is observed that the phase shifts $\Delta P$ is high for the planar geometry, intermediate for cylindrical geometry, while it is low for the spherical geometry. If we increase $\mu_1$ to very large values the nonplanar geometries would approach the planar geometry. Therefore, the nature and the collision of
IASWs significantly change for large values of $\mu_1$, the scope of the present work.

![Graph showing the variation of $\Delta P$ with $\kappa$ for planar geometry ($v = 0$), cylindrical geometry ($v = 1$) and for spherical geometry ($v = 2$) at $\varphi_{\kappa} = 0.9, \varphi_{\eta} = 0.3, \epsilon = 0.1, \varphi_{\eta} = 0.1, \varphi_{\eta} = 0.4$]

Figure 3 presents the variation of the phase shift $\Delta P$ against the superthermal parameter $\kappa$ for at the different values of $v = 0, 1$ and 2. For a given $v$, the phase shift $\Delta P$ decreases rapidly with $\kappa$ when $\kappa \leq 2.5$ and increases smoothly with $\kappa$ when $\kappa \geq 2.5$. For a given value of $\kappa$, the magnitude of the phase shift $\Delta P$ for the planar geometry is more than its value for the nonplanar geometry.

![Graph showing the variation of $\Delta P$ with $\theta$ for compressive ($\mu = 2$) and rarefactive ($\mu = 10$) at ($v = 0$) for planar geometry, ($v = 1$) for cylindrical geometry, and ($v = 2$) for spherical geometry]

Fig. 4. The variation of $\Delta P$ with $\theta$ for compressive ($\mu = 2$) and rarefactive ($\mu = 10$) were ($v = 0$) for planar geometry, ($v = 1$) for cylindrical geometry, and ($v = 2$) for spherical geometry.
In Fig. 4 we have plotted the phase shifts variation against $\theta_1$ for planar and nonplanar geometry. It is observed that: for a given value of $v$, the phase shift $\Delta P$ increases with $\theta_1$. For a given value of $\theta_1$, the phase shift curves of the cylindrical geometry and the spherical geometry are fairly close to each other, whereas the phase shift curve for the planar geometry are comparatively far away from them.

For planar, cylindrical and spherical geometries, Figs. 5 and 6 show the variation of the negative ion-to-positive ion density ratio against the space coordinate $x$ and the time variable $t$, respectively. Additionally, Figures 5 and 6 demonstrate that two rarefactive and compressive IASWs propagate in opposite directions approach to each other, collide, and asymptotically separate away, respectively. We observe that during collision one practically motionless composite structure forms for some time interval.
After the interaction, one can note that the trajectories of the rarefactive and compressive IASWs have deviated from the initial trajectories. In fact, these deviations are just the phase shifts for two IASWs [33]. Physically, this means that the negative ion-to-positive ion density ratio has an important effect on the dynamical behavior (i.e., rarefactive or compressive) of IASWs. This agrees with result obtained by El-Labany et al. [43].

IV. SOLITONS AND PHASE SHIFTS AT CRITICAL NEGATIVE IONS DENISTY:

In this section we take the ion densities to be critical \( \mu_2c \), with \( A = 0 \). Without loss of generality, the quadratic nonlinearity in the KdV equations 13-14 disappears, and \( \phi_2\xi = 0 \) and \( \phi_2\eta = 0 \), but \( \phi_2 \neq 0 \), and we must keep the contributions in the first order perturbation solutions. Accordingly, Eq. 10 becomes

\[
n_l^{(2)} = \frac{3\beta^2}{2\lambda^4}(\phi_1^2 + \phi_2^2) \quad (27a) \quad u_+^{(2)} = \frac{1}{2\lambda^3}(\phi_1^2 - \phi_2^2), \quad (27c)
\]
where $\delta_1 \neq 0$.

At critical negative ion density, the third order variables becomes

\begin{align*}
\lambda \tilde{T}_{n_+}^{(3)} + \tilde{T}_{n_+}^{(1)} + \frac{v}{R} \tilde{u}_+^{(1)} + \frac{\lambda}{R} \tilde{u}_+^{(3)} + \tilde{R}\psi_3 &= 0, \\
\lambda \tilde{T}_{n_-}^{(3)} + \tilde{T}_{n_-}^{(1)} + \frac{v}{R} \tilde{u}_-^{(1)} + \frac{\lambda}{R} \tilde{u}_-^{(3)} - \beta \tilde{R}\psi_3 &= 0.
\end{align*}

(28a) \hspace{1cm} (28b)

\begin{align*}
\left(\frac{\partial^2}{\partial \xi^2} + 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) \phi_1 - \mu_1 &- \frac{k - \frac{1}{2}}{\lambda} \phi_3 \\
- \mu_1 &\left(\frac{k - \frac{1}{2}}{\lambda} + \frac{1}{2}\right)^2 \phi_1^3 - \mu_2 n_+^{(3)} + n_-^{(3)} = 0.
\end{align*}

(28c) \hspace{1cm} (28d)

(28e)

Solving equations containing the terms depending on $\xi$ or $\eta$, besides $\tau$ we obtain the modified cylindrical Korteweg-de Vries (mKdV) equations.

\begin{align*}
\frac{\partial \phi_{1\xi}}{\partial R} + C \phi_{1\xi}^2 \frac{\partial \phi_{1\xi}}{\partial \xi} + B \frac{\partial^3 \phi_{1\xi}}{\partial \xi^3} + \frac{v}{2R} \phi_{1\xi} &= 0, \\
\frac{\partial \phi_{1\eta}}{\partial R} + C \phi_{1\eta}^2 \frac{\partial \phi_{1\eta}}{\partial \eta} + B \frac{\partial^3 \phi_{1\eta}}{\partial \eta^3} + \frac{v}{2R} \phi_{1\eta} &= 0. \\
\frac{\partial \phi_{1\xi}}{\partial R} &= -E \phi_{1\xi}, \\
\frac{\partial \phi_{1\eta}}{\partial R} &= -E \phi_{1\eta},
\end{align*}

(29a) \hspace{1cm} (29b) \hspace{1cm} (29c) \hspace{1cm} (29d)

where $C = \frac{15(1+\mu_2\beta^3)}{4\lambda^4(1+\mu_2\beta)}$, and $E = \frac{(1+\mu_2\beta^3)}{8\lambda^4(1+\mu_2\beta)}$.

Because mKdV equations like 29a and 29b are invariant for a sign inversion of $\phi_{1\xi}$ or $\phi_{1\eta}$ the one-soliton solutions can be written as

\begin{align*}
\phi_{1\xi} &= \pm \phi_{0\xi} \theta_{1\xi} \operatorname{sech} \left( \phi_{0\xi} \sqrt{\frac{C}{6B}} \theta_{1\xi} \left[ \xi - \frac{1}{6} C \phi_{0\xi}^2 \theta_{1\xi}^3 \right] \right), \\
\phi_{1\eta} &= \pm \phi_{0\eta} \theta_{1\eta} \operatorname{sech} \left( \phi_{0\eta} \sqrt{\frac{C}{6B}} \theta_{1\eta} \left[ \eta - \frac{1}{6} C \phi_{0\eta}^2 \theta_{1\eta}^3 \right] \right).
\end{align*}

(30) \hspace{1cm} (31)

where $\phi_{0\xi}$ and $\phi_{0\eta}$ are the maximum amplitudes of IASWs in their initial position. Note that the respective $\pm$ signs are not coupled. After the same
mathematical strategy that is used in Sec. III, we finally obtain the corresponding phase shifts after collision of the two IASWs in the critical case.

\[ \Delta P_c = 2\varepsilon^2 E \Phi_0 \left( \frac{6B}{C} \right)^{1/2} \theta^y_z, \]
\[ \Delta Q_c = -2\varepsilon^2 E \Phi_0 \xi \left( \frac{6B}{C} \right)^{1/2} \theta^y_z. \]

V. CONCLUSIONS

Finally, we conclude the critical case, it is important to mention that the interaction of the IASWs in planar one-dimensional geometry, cylindrical, and spherical geometry are different.
Fig. 8 In the critical case, the contour plot for the colliding process of two dark/bright IASWs at the cylindrical case $v = 1$ for the same values as in fig. 10 at different times a) $t = 0$, b) $t = 42$ and c) $t = 75$.

Figures 7(a)–7(c) [8(a)–8(c) and 9(a)–9(c)] represents the contour plot for the colliding processes between two bright/dark planar [cylindrical and spherical] IASWs at different times which admits combination of bright and dark IASWs collision agrees with [43]. It’s clear that the two bright/dark of planar and nonplanar IASWs interact to create single composite structure at different times. This behavior may explain as follows: the wave speed for planar IASWs is greater than the cylindrical and the spherical IASWs and cylindrical IASWs are much greater than spherical IASWs.
Fig. 9. In the critical case, two dark/bright IASWs at the spherical case \( \nu = 2 \) for the same values as in fig. 10 at different times a) \( t = 0 \), b) \( t = 50 \) and c) \( t = 80 \).

Fig. 10. The variation of the phase shift \( \Delta P_c \) with \( \theta \) (\( \nu = 0 \)) for planar geometry, (\( \nu = 1 \)) for cylindrical geometry, and (\( \nu = 2 \)) for spherical geometry.
Fig. 10 explores the difference in phase shifts $\Delta P_c$ for planar and nonplanar geometries at the critical case. If the angle $\theta \leq 10$ the planar geometry approach the nonplanar geometries but when $\theta \geq 10$ they became to be far away from each other.

Fig. 11, The variation $\Delta P_c$ with $\kappa$ ($v = 0$) for planar geometry, ($v = 1$) for cylindrical geometry, and ($v = 2$) for spherical geometry.

Fig. 11 represents the phase shifts change the superthermal parameter $\kappa$ for planar and nonplanar (cylindrical, and spherical) geometry. For a given value of $v$, the phase shift curves decreases rabidly until $\kappa=5$ for the planar geometry and almost $\kappa=3$ for the cylindrical geometry and the spherical geometry. Figures 12 show the variations of the negative ion-to-positive ion density ratio against the space coordinate $x$ and the time variable $t$, respectively, for the critical case.

Continuously, Figs 12 ensure that the compressive IASWs propagate in opposite directions approach to each other, collide, and asymptotically separate away, respectively. After the interaction, one can note that the trajectories of compressive IASWs have deviated from the initial trajectories for the planar and the nonplanar geometry. This agrees with result obtained [43].
In this manuscript we considered a multicomponent plasma consisting of mobile positive and negative ions, and superthermal electrons in planar and nonplanar geometry. Accordingly, we employed the extended PLK method, the nonlinear mKdV equations and their change in trajectories were derived. For the generic case, the nonlinear propagation and the collision of IASWs are described by nonlinear cKdV equations and their corresponding phase shifts. Furthermore, the effects of some plasma parameters such as; the negative ion-to-positive ion density ratio \( \mu_2 \), superthermal parameter \( \kappa \), and the positions of IASWs on dynamics and collisions of IASWs are discussed and graphically displayed. In particular, it is found that negative ion-to-positive ion density ratio play an important role in the planar and nonplanar geometry not only on the formations and the dynamical behavior of IASWs, but also on the IASWs collision which translated in the presented phase shifts.
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